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LETTER TO THE EDITOR

Dynamical properties of quasi-crystals: Fibonacci chain and Penrose lattice

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Abstract. The discrete scale invariance of quasi-periodic systems leads to a scaling relationship $\Omega = L^{-z}f(L)$ between a characteristic dynamic variable Ω and length L, where $f(L)$ is a periodic function of $\ln L$. The dynamic exponent z can be calculated for diffusion, spin wave and phonon dynamics (where Ω is respectively Γ, ω, ω^2 where Γ and ω are characteristic rates or frequencies respectively) by exploiting a crossover argument which results in $z = d_f + \hat{t}$ where d_f is the fractal dimension and $\hat{t}$ is the length scaling exponent for the conductance. The latter exponents and hence z, are calculated for a Fibonacci chain model $((d_f, \hat{t}, z) = (1, 1, 2))$ and, via an exact bond moving technique, for the Penrose lattice $((d_f, \hat{t}, z) = (2, 0, 2))$. The technique also provides the thermal exponent of the Heisenberg spin model on the two quasi-crystals ($\nu = 1, \infty$, respectively).

Static and dynamic properties can become anomalous because of the scale invariance (Kadanoff 1966, Wilson 1971) occurring at thermal or geometric transitions (see, for example, Stinchcombe 1985a), where a correlation length diverges, or in the incommensurate limit, in which the size of a periodically repeated basic cell diverges (see, for example, Bak 1982). Quasi-periodic systems, such as the Fibonacci chain (Levine and Steinhardt 1984) and Penrose lattice (Penrose 1979) treated here, provide interesting models, because of their simple hierarchical nature, of a diverging basic cell, and hence of incommensurate behaviour. They have also received much current attention as models of quasi-crystals, of which real examples appear to have been found (Shechtman *et al* 1984).

Dynamical properties of the simplest quasi-periodic system, the Fibonacci chain, have been discussed by Luck and Petritis (1986), and by Bell (1986), following earlier work on a related quasi-periodic Schrödinger problem by Kohmoto *et al* (1983) using a recursion relation for the trace of a transfer matrix. An alternative approach by Khantha and Stinchcombe (1987) treats diffusion on the Fibonacci chain using exact scaling equations for the 'waiting time' distribution, yielding $z = 2$ for the dynamic exponent, consistent with numerical work by Luck and Petritis (1986) for the density of states in their related problem.

Dynamical properties of the higher dimensional ($d \geq 2$) quasi-periodic systems are much more difficult to treat: until now there has been a discussion, using Conway's theorem, of the extended against localised nature of states in the Penrose lattice (Tsunetsugo *et al* 1985) and, very recently, numerical work on the tight-binding electronic spectrum (Ogadaki and Nguyen 1986). The aim of the present letter is to point out certain general features and inter-relationships and then to provide an analytic approach to dynamical properties of quasi-periodic systems which yields in particular

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the dynamic exponent for the two-dimensional Penrose lattice; the approach is of more general applicability, for example giving very simply the previously obtained result for the dynamic exponent of the Fibonacci chain.

The method expoits the hierarchical construction of the quasi-periodic system under consideration, using decimation to work backwards along the hierarchy. The recursive construction of the hierarchy is simple for the Fibonacci chain (Levine and Steinhardt 1984) and has also been given in a convenient way by Robinson (1975) for the more difficult case of the Penrose lattice. The philosophy of reversing Robinson's construction has recently been used by Godrèche *et al* (1986) to obtain thermal properties of Ising spins on the Penrose lattice, in an approximation using bond moving (Migdal 1975, Kadanoff 1976). A similar procedure is used here, except that for our considerations bond moving is exact.

Such recursively generated systems have a 'discrete' scale invariance which leads, as will be explained, to an interesting periodicity in the dynamical (and other) behaviour. Our approach to the dynamics relies on a series of connections, given previously in other contexts, which relate the behaviour in an asymptotic critical regime via a crossover argument to the length scaling of a static quantity analogous to a diffusion constant; that in turn is related by an Einstein relation to the scaling of conductance and density, to which the decimation and bond-moving technique can be relatively simply applied. Because the conductance scaling is the same as the scaling of a Heisenberg model thermal variable (Stinchcombe 1979) the technique also yields thermal exponents for the Heisenberg spin system on the quasi-crystals (Fibonacci chain and Penrose lattice), as well as the conductance exponent.

The Fibonacci chain is constructed from two line segments ('tiles') A_0, B_0, by the recursively applied rules (Levine and Steinhardt 1984)

$$A_{n+1} = A_n B_n \qquad B_{n+1} = A_n \tag{1}$$

where AB denotes the line segment obtained by adding A and B end-to-end with A to the left of B. The line segment length ratio A_n/B_n is the same for all n if it takes the 'Golden mean' value $\tau = \frac{1}{2}(1+\sqrt{5})$. The resulting system is then scale invariant under discrete dilatations with length scale factor $b = \tau$. A similar recursive construction of the Penrose lattice has been given by Robinson (1975) (see Grünbaum and Shephard 1986, Godrèche *et al* 1986) using two triangular tiles with coloured (black or white) vertices, combined according to auxiliary matching rules. Each tile is reproduced, with colour reversal and dilation by length scale factor $b = \tau$, by combining two tiles from the previous stage of construction. Two such steps thus take the lattice into a scaled version of itself. Both the Fibonacci and Penrose lattices thus possess 'discrete' scale invariance (limited to a special scale factor b $(= \tau, \tau^2)$ or any integer power of that b).

The dynamic and other processes considered here will arise from the coupling of sites of the quasi-periodic lattices by bonds corresponding to the two line segments of the Fibonacci chain or the four distinct types of edge of Robinson tiles in the Penrose lattice (a fuller description of these edges is given in Godrèche *et al* (1986)). For phonon and magnon dynamics, and diffusion processes, the bonds represent spring, exchange and hopping constants respectively. These three processes are governed by similar basic equations of motion, and so can be discussed together using the dynamic variable

$$\Omega = \omega^2, \omega, \Gamma \tag{2}$$

for phonons and magnons (where ω is the characteristic frequency) and diffusion (where Γ is the characteristic rate), respectively.

It can be shown that, because of the discrete scale invariance (scale factor b), the length scaling behaviour of the characteristic dynamic variable is, in general, for small Ω and large length L

$$\Omega(L) = L^{-z} f(L) \tag{3}$$

where z is the dynamic exponent (to be determined) and $f(L)$ is a periodic function of $\ln L$ with period $\ln b$. The result (3) follows because explicit scaling of systems with the discrete scale invariance can at most relate $\Omega(bL)$ to $\Omega(L)$. The ratio of these quantities for small Ω (i.e. large L) is the eigenvalue λ of a linearised renormalisation group equation (cf (15)):

$$\Omega(bL)/\Omega(L) = \lambda \equiv b^{-z}. \tag{4}$$

The last step can be taken to define z. Equation (3) is consistent with (4) provided first that the z determined by (4) is the dynamic exponent in (3), and secondly that $f(Lb)$ is the same as $f(L)$; from this the periodicity property stated under (3) follows, for example by considering the function y defined by $y\ (\ln L) \equiv f(L)$. The above argument is of course not limited to dynamic quantities, and related viewpoints have previously been given for thermal properties (Nauenberg 1975), particularly in Berker lattices (Derrida *et al* 1984) and other fractals (Stinchcombe 1985b). A further consequence is that a general dynamic quantity $P(\Omega)$ of hierarchically constructed systems, such as the Berker or Sierpinski gasket fractals (Mandelbrot 1977) or the Fibonacci chain or Penrose lattice, should for small frequencies behave like

$$P(\Omega) = \Omega^{x} F(\Omega) \tag{5}$$

where F is a periodic function of $\ln \Omega$ with period $z \ln b$. Such behaviour has indeed been seen in the numerical investigations by Maggs and Stinchcombe (1986) of the dynamic response of the Sierpinski gasket fractal, and by Luck and Petritis (1986) of the density of states of the Fibonacci chain.

Instead of obtaining the dynamic exponent z by directly scaling the dynamic variable Ω, as indicated in (4), we may get it by scaling static quantities whose exponents are related to z by the following crossover argument, similar to ones given previously for critical anomalies produced by a diverging correlation length (Harris and Stinchcombe 1983, Aharony 1985). We consider the composite 'tile' obtained at the nth stage of construction of, for example, the Fibonacci chain. This 'tile', of length $L_n \propto b^n$, is then periodically repeated to obtain a system (typical of that considered in numerical work on the Fibonacci chain) whose dynamics at long 'wave' length L $(L \gg L_n)$ is characterised by dynamic variable Ω with the 'normal' dependence

$$\Omega = D(L_n) L^{-2} \qquad (L \gg L_n) \tag{6}$$

where D is an effective diffusion constant, spin wave stiffness, or square of velocity for sound, depending on the situation considered; it is convenient to use the diffusion case, the others being analogous. We are, however, interested in the anomalous dependence $\Omega \propto L^{-z}$ which according to (3) obtains in the limit $L_n \to \infty$ at fixed (and large) L. (For this argument we suppress the $f(L)$ factor in (3), though it is strictly present and can be exhibited with only a slight lengthening of equations). The two limits $L \gg L_n$ and $L \ll L_n$ are included in the scaling form

$$\Omega = L^{-z} g(L/L_n) \tag{7}$$

where the scaling function has the asymptotic dependences

$$g(x) \sim \text{constant} \qquad x \ll 1$$
$$\sim x^{z-2} \qquad x \gg 1. \tag{8}$$

As a consequence, the dependence on L_n of the diffusion constant in (6) has to be

$$D(L_n) \propto L_n^{2-z} \tag{9}$$

involving the dynamic exponent. z can therefore be obtained from the n dependence of the static quantity $D(L_n)$, i.e. from the scaling of the diffusion constant between two successive members n, $n+1$ of the Fibonacci chain hierarchy. The same discussion applies also to the Penrose lattice or any similar hierarchically generated system.

It is helpful to make a further connection, in the spirit of earlier work (Kirkpatrick 1979, Harris and Stinchcombe 1983, Aharony 1985), by using the Einstein formula to relate the diffusion constant D to the conductivity Σ and density ρ:

$$D(L_n) = \frac{\Sigma(L_n)}{\rho(L_n)} = \frac{L_n^{-\tilde{t}}}{L_n^{-\tilde{\beta}}}. \tag{10}$$

$\tilde{t}$ and $\tilde{\beta}$ are length scaling exponents for conductivity and density. Comparison with (9) yields

$$z = 2 + \tilde{t} - \tilde{\beta} \tag{11}$$

relating z and the static exponents $\tilde{t}$, $\tilde{\beta}$. In turn $\tilde{\beta}$ is related to the space and fractal (mass) dimensions d, d_f by $\tilde{\beta} = d - d_f$ (see, for example, Stinchcombe 1985b), and $\tilde{t}$ can be obtained from the conductance length scaling exponent $\hat{t}$ from which it differs by $(d-2)$, so an alternative version of (11) is

$$z = d_f + \hat{t}. \tag{12}$$

A final relationship follows from the exact equivalence (Stinchcombe 1979, Coniglio 1981) between the scalings of conductance σ and of the thermal variable $D = J/K_B T$ of the Heisenberg model at low temperatures (J is the exchange constant). This implies that the correlation length exponent ν is

$$\nu = 1/\hat{t}. \tag{13}$$

Thus by scaling the conductance it is possible to obtain both the static thermal exponent of the Heisenberg model and the dynamic exponent z provided the fractal dimension d_f is known.

For both Fibonacci chain and Penrose lattice the fractal dimension is trivially the same as the space dimension ($d_f = d$) because both lattices are space filling. It remains to carry out the conductance scaling, and this is accomplished by procedures analogous to those in Stinchcombe and Watson (1976).

For the Fibonacci chain the conductance scaling for length scale factor $b = \tau$ is simply given by the series combination of two conductances σ_n, σ_{n-1} (corresponding to A_n and $B_n = A_{n-1}$), resulting from decimating the intermediate site to arrive at the conductance corresponding to A_{n+1} in (1):

$$\frac{1}{\sigma_{n+1}} = \frac{1}{\sigma_n} + \frac{1}{\sigma_{n-1}}. \tag{14}$$

In the scale-invariant limit where $\sigma_{n+1}/\sigma_n \equiv \sigma(L_{n+1})/\sigma(L_n)$ is independent of n this gives 'eigenvalue' (in the same sense as in equation (4)):

$$1/\tau = \sigma(L_{n+1})/\sigma(L_n) \equiv (L_{n+1}/L_n)^{-\hat{t}} = b^{-\hat{t}}. \qquad (15)$$

So $\hat{t}$ is unity and (12), (13) yield

$$z = 2 \qquad \nu = 1 \qquad (\hat{t} = 1, d_f = 1) \qquad (16)$$

for dynamic and Heisenberg thermal exponents of the Fibonacci chain. The value of z is in agreement with that found by Khantha and Stinchcombe (1987) by treating the particular case of diffusion.

For the Penrose lattice the explicit scaling is more subtle, but it can be carried out using bond moving, which is exact in the low-temperature limit (Migdal 1975) for the thermal variable K which determines the (Gaussian) fluctuations of the Heisenberg model, and hence (by their one-to-one correspondence) also for the conductance. The bond-moving and decimation steps are relatively easy to identify in the Robinson construction of the lattice and, in the subscript notation of Godrèche *et al* (1986), they correspond to transforming four variables y_i $(i = 1, \ldots, 4)$ which combine under a mapping $\mathscr{T}$ (whose square is the full recursion mapping of the Penrose lattice onto itself) according to

$$\begin{aligned} &y'_1 = y_4, y'_2 = y_3, y'_3 = S(P(y_1, y_3), P(y_3, y_4)) \\ &y'_4 = S(P(y_1, y_2), P(y_1, y_4)). \end{aligned} \qquad (17)$$

Here the prime denotes a scaled variable (i.e. $y \to y'$ is like increasing by unity the label n indicating the number of recursive construction steps); $S(y_i, y_j)$ denotes a decimation combination of two bonds y_i, y_j, and $P(y_i, y_j)$ denotes a bond-moving combination. In our case the variables y_i are conductances, $y_i = \sigma_i$, and so the decimation and bond-moving combinations are respectively series and parallel combinations:

$$S(\sigma_i; \sigma_j) = \sigma_i\sigma_j/(\sigma_i + \sigma_j) \qquad P(\sigma_i, \sigma_j) = \sigma_i + \sigma_j. \qquad (18)$$

The transformation (17) is associated with length scale change $L_{n+1}/L_n = \tau$.

The scale invariant situation (i.e. the fixed point of (17)) is $\sigma_1^* = \sigma_2^* = \sigma_3^* = \sigma_4^*$ and linearisation around the fixed point yields eigenvalue $\lambda = 1$ for the conductance scale factor accompanying length scale factor $b = \tau$. Hence $\hat{t}$ is zero and (12), (13) give for the Penrose lattice

$$z = 2 \qquad \nu = \infty \qquad (\hat{t} = 0, d_f = 2). \qquad (19)$$

The infinite value of the exponent ν for the Heisenberg correlation length arises from the marginal scaling $(\lambda = 1)$ of the thermal variable K, and implies that in the Penrose lattice the correlation length is (apart from periodic dependences) a power of exp K, as in the two-dimensional Heisenberg model (Migdal 1975).

For both the Penrose lattice and the Fibonacci chain the value $z = 2$ was found for the dynamic exponent (for the dynamic variable Ω defined in (2)). It should be emphasised that this result cannot be explained by a continuum argument (as applies in simpler situations like (6)) since in the situation where the defining equation (3) for z applies, the scale of the repeating tile has diverged. A further point is that periodicities still remain in the dynamics in the situation we discuss. A consequence of (16), (19) is that the accumulated numbers of states up to Ω for Fibonacci and Penrose lattices have form (5) with $x(= d_f/z) = \frac{1}{2}$, 1 respectively. The Fibonacci chain result agrees with the numerical work of Luck and Petritis (1986).

The present method is not able to provide details of the periodic functions, though it allows for their existence. It would be interesting to have numerical investigations for the Penrose lattice for any of the dynamic processes here considered (phonons, magnons, diffusion). Further work on the thermal behaviour of the low-temperature Heisenberg model on Fibonacci or Penrose lattices would also be desirable. Apart from features accessible numerically, frustration effects and low-temperature free energies could be investigated by the analytic method we have provided, along the lines of the work of Godrèche *et al* (1986) for the Ising model.

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References

Aharony A 1985 *Scaling Phenomena in Disordered Systems* ed R Pynn and A Skjeltorp (New York: Plenum) p 289
Bak P 1982 *Rep. Prog. Phys.* **45** 587
Bell S C 1986 *D Phil Thesis* Oxford University
Coniglio A 1981 *Phys. Rev. Lett.* **46** 250
Derrida B, Itzykson C and Luck J M 1984 *Commun. Math. Phys.* **94** 115
Godrèche C, Luck J M and Orland H 1986 *Saclay Preprint*
Grünbaum B and Shephard G C 1986 *Tiling and Patterns* (San Francisco: Freeman)
Harris C K and Stinchcombe R B 1983 *Phys. Rev. Lett.* **50** 1399
Kadanoff L P 1966 *Physics* **2** 263
—— 1976 *Ann. Phys., NY* **100** 359
Khantha M and Stinchcombe R B 1987 *J. Phys. A: Math. Gen.* **20** 495
Kirkpatrick S 1979 *Ill-condensed Matter* ed R Balian, R Maynard and G Toulouse (Amsterdam: North-Holland) p 321
Kohmoto M, Kadanoff L P and Tang C 1983 *Phys. Rev. Lett.* **50** 1870
Levine D and Steinhardt P J 1984 *Phys. Rev. Lett.* **53** 2477
Luck J M and Petritis D 1986 *J. Stat. Phys.* to be published
Maggs A C and Stinchcombe R B 1986 *J. Phys. A: Math. Gen.* **19** 2637
Mandlebrot B 1977 *Fractals* (San Francisco: Freeman)
Migdal A A 1975 *Sov. Phys.-JETP* **42** 743
Nauenberg M 1975 *J. Phys. A: Math. Gen.* **8** 925
Ogadaki T and Nguyen D 1986 *Phys. Rev.* B **33** 2184
Penrose R 1979 *Math. Intell.* **2** 32
Robinson B 1975 *University of California Preprint*
Shechtman B, Blech I, Gratias D and Cahn J W 1984 *Phys. Rev. Lett.* **53** 1951
Stinchcombe R B 1979 *J. Phys. C: Solid State Phys.* **12** L389, 2625
—— 1985a *Highlights of Condensed Matter Physics* ed F Bassani, F Fumi and M P Tosi (Amsterdam: North-Holland) p 849
—— 1985b *Scaling Phenomena in Disordered Systems* ed R Pynn and A Skjeltorp (New York: Plenum) pp 13, 465
Stinchcombe R B and Watson B P 1976 *J. Phys. C: Solid State Phys.* **9** 3221
Tsunetsugo H, Fujiwara T, Kazuo U and Tokihiro T 1985 *University of Tokyo preprint*
Wilson K G 1971 *Phys. Rev.* B **4** 3174